

# Gromov-Witten, Gopakumar-Vafa, and Donaldson-Thomas invariants of Calabi-Yau threefolds

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*Abstract:* Gromov-Witten, Gopakumar-Vafa, and Donaldson-Thomas invariants of Calabi-Yau threefolds are compared. In certain situations, the Donaldson-Thomas invariants are very easy to handle, sometimes easier than the other invariants. This point is illustrated in several ways, especially by revisiting computations of Gopakumar-Vafa invariants by Katz, Klemm, and Vafa in a rigorous mathematical framework. This note is based on my talk at the 2004 Snowbird Conference on String Geometry.

## 1 DT Invariants and GW Invariants

### 1.1 Generalities

Let  $X$  be a nonsingular complex projective threefold,  $\beta \in H_2(X, \mathbf{Z})$ , and let  $n \in \mathbf{Z}$ . We let  $I_n(X, \beta)$  denote the part of the Hilbert scheme of  $X$  parametrizing subschemes  $Z \subset X$  with

- $[Z] = \beta$
- $\chi(\mathcal{O}_Z) = n$ .

The class  $[Z] \in H^2(X, \mathbf{Z})$  can be equivalently defined as either the dimension one component of the support cycle of  $Z$ , or as  $\text{ch}_2(\mathcal{O}_Z)$ .

We let  $\mathcal{I}_Z$  be the ideal sheaf of  $Z$ . In [1], a perfect obstruction theory is defined on  $I_n(X, \beta)$  arising naturally from the deformation theory of the ideal sheaves  $\mathcal{I}_Z$ . The virtual dimension is given by

$$D = \dim \text{Ext}_0^1(\mathcal{I}_Z, \mathcal{I}_Z) - \dim \text{Ext}_0^2(\mathcal{I}_Z, \mathcal{I}_Z) = c_1(X) \cdot \beta. \quad (1)$$

In (1), the subscript of 0 denotes traceless Ext, i.e. the kernel of the trace map:

$$\mathrm{Ext}_0^i(\mathcal{I}_Z, \mathcal{I}_Z) = \ker \left( \mathrm{Ext}^i(\mathcal{I}_Z, \mathcal{I}_Z) \rightarrow H^i(X, \mathcal{O}) \right). \quad (2)$$

Associated to the perfect obstruction theory, there is a virtual fundamental class

$$[I_n(X, \beta)]^{\mathrm{vir}} \in H_{2D}(I_n(X, \beta)). \quad (3)$$

If  $\beta = 0$ , then the virtual dimension  $D$  is 0 for all  $n$ . Putting

$$D_0^n = \deg[I_n(X, 0)]^{\mathrm{vir}} \in \mathbf{Z} \quad (4)$$

and introducing a formal variable  $q$ , the *dimension 0 Donaldson-Thomas partition function* is defined as the formal series

$$Z_0^{\mathrm{DT}}(X) = \sum_{n=0}^{\infty} D_0^n q^n. \quad (5)$$

**Conjecture 1 ([MNOP1])**

$$Z_0^{\mathrm{DT}}(X) = M(-q)^{\int_X c_3 - c_1 c_2}.$$

Here  $M(q)$  is the McMahon function, the generating function of three dimensional partitions,

$$M(q) = \prod_{n=1}^{\infty} (1 - q^n)^{-n}. \quad (6)$$

The  $c_i$  are the Chern classes of  $X$ . In particular,  $\int_X c_3 = e(X)$ , the topological Euler characteristic of  $X$ .

Conjecture 1 has been proven if  $X$  is a toric variety [MNOP2]. A key feature is the use of localization which relates the conjecture to a combinatorial counting of 3 dimensional partitions. The relevance of 3 dimensional partitions to Gromov-Witten theory appeared in [ORV]. In the present note, we will explain some simple computations in DT theory which do not rely on localization.

For each  $n$ , Conjecture 1 makes a prediction for  $D_0^n$ . This prediction is trivially true for  $n = 1$  and has been proven for  $n = 2$  by Maulik and Pandharipande [P3]. A simpler argument can be given in the Calabi-Yau case for  $n \leq 3$  as will be described below.

For  $\beta \neq 0$ , the virtual dimension need not be zero. Nevertheless, invariants can be obtained by imposing additional conditions [MNOP2] or by working equivariantly [BP].

Note that another important way to achieve virtual dimension  $D = 0$  for all  $n$  (and all  $\beta$  as well) is if  $X$  is Calabi-Yau, which we take in this paper to simply mean that  $K_X$  is trivial. This case is the focus of the present paper, and from now on we assume that  $X$  is Calabi-Yau. Note that the conclusion on the virtual dimension also holds more generally if  $c_1(X)$  is torsion.

## 1.2 Calabi-Yau case

Assuming that  $X$  is Calabi-Yau, we now put

$$D_\beta^n = \deg[I_n(X, \beta)]^{\text{vir}} \in \mathbf{Z} \quad (7)$$

and define the *degree  $\beta$  Donaldson-Thomas partition function* by

$$Z_\beta^{\text{DT}}(X) = \sum_n D_\beta^n q^n. \quad (8)$$

We now introduce formal symbols  $t^\beta$  satisfying  $t^\beta t^{\beta'} = t^{\beta+\beta'}$  and define the full *Donaldson-Thomas partition function*

$$Z^{\text{DT}}(X) = \sum_\beta Z_\beta^{\text{DT}}(X) t^\beta \quad (9)$$

and the *reduced Donaldson-Thomas partition function*  $Z^{\text{DT}}(X)'$  and *reduced degree  $\beta$  Donaldson-Thomas partition functions*  $Z_\beta^{\text{DT}}(X)'$  by

$$Z^{\text{DT}}(X)' = \frac{Z^{\text{DT}}(X)}{Z_0^{\text{DT}}(X)} =: \sum_\beta Z_\beta^{\text{DT}}(X)' t^\beta. \quad (10)$$

It is then natural to define the *reduced Donaldson-Thomas invariants*  $D_\beta^{n'}$  by

$$Z_\beta^{\text{DT}}(X)' = \sum_n D_\beta^{n'} q^n. \quad (11)$$

Now recall the Gromov-Witten invariants  $N_\beta^g = \deg[\overline{M}_{g,0}(X, \beta)]^{\text{vir}} \in \mathbf{Q}$ . Form the generating functions

$$F'_g(X) = \sum_{\beta \neq 0} N_\beta^g t^\beta, \quad F'(X) = \sum_g \lambda^{2g-2} F'_g, \quad Z^{\text{GW}}(X)' = \exp(F'(X)).$$

The prime is used to emphasize that the constant maps  $\beta = 0$  are not included.

**Conjecture 2** ([MNOP1])  $Z^{\text{DT}}(X)' = Z^{\text{GW}}(X)'$  after the change of variables  $q = -e^{i\lambda}$ .

In [MNOP1], localization techniques were used to prove Conjecture 2 if  $X$  is a noncompact toric Calabi-Yau threefold, under the additional assumption that the topological vertex prediction [AKMV, ORV] correctly computes the Gromov-Witten partition function.

Note that in [MNOP1], a compactification of  $X$  is used to make sense of the Donaldson-Thomas invariants of  $X$ , and then  $Z^{\text{DT}}(X)'$ , appropriately defined, is independent of the choice of compactification. The invariants can also be defined in these non-compact cases by working equivariantly as in [BP].

Donaldson-Thomas invariants are often easy to compute in the Calabi-Yau case. Suppose  $I_n(X, \beta)$  is smooth. Then for  $\mathcal{I} \in I_n(X, \beta)$ , we have  $\text{Ext}_0^1(\mathcal{I}, \mathcal{I}) = T_{I_n(X, \beta), \mathcal{I}}$ . Since  $\text{Ext}_0^2(\mathcal{I}, \mathcal{I}) \simeq (\text{Ext}_0^1(\mathcal{I}, \mathcal{I}))^*$  canonically (up to a global constant), it follows that the obstruction bundle is  $T^*I_n(X, \beta)$ . If furthermore we have a decomposition

$$I_n(X, \beta) = \cup M_i$$

into connected components, then we have the simple formula

$$D_\beta^n = \sum_i e(T^*M_i) = \sum_i (-1)^{\dim(M_i)} e(M_i). \quad (12)$$

where  $e(M_i)$  denotes the topological Euler characteristic of  $M_i$ .

Note that in the Calabi-Yau case, Conjecture 1 takes the simpler form

$$Z_0^{\text{DT}}(X) = M(-q)^{e(X)}. \quad (13)$$

Compare (1) with the following.

**Proposition 1** ([C])  $\sum_{n=0}^\infty e(I_n(X, 0))q^n = M(q)^{e(X)}$ .

Note that  $I_n(X, 0)$  is smooth for  $n \leq 3$  and singular for  $n \geq 4$ .<sup>1</sup> It follows that for  $n \leq 3$  we have  $D_0^n = (-1)^{3n} e(I_n(X, 0)) = (-1)^n e(I_n(X, 0))$ . It

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<sup>1</sup>The singular locus of  $I_4(X, 0)$  is described in e.g. [K].

follows immediately from (12) and (1) that  $Z_0^{\text{DT}}(X) \equiv M(-q)^{e(X)} \pmod{q^4}$ . The same argument shows that (13) is equivalent to

$$D_0^n = (-1)^n e(I_n(X, 0))$$

for all  $n$ .

As a simple illustration of the underlying ideas, we explain why  $D_0^3$  is the coefficient of  $q^3$  in  $M(-q)^{e(X)}$ .

First note that  $I_3(X, 0)$  is the Hilbert scheme of degree three zero dimensional subschemes in  $X$ , which is well known to be smooth. Next,  $I_3(X, 0)$  can be obtained from  $\text{Sym}^3 X$  by two blowups. First blow up along the small diagonal  $\Delta_s$ , which is isomorphic to  $X$ . Then blow up the proper transform of the big diagonal  $\Delta$  to get  $I_3(X, 0)$ . We have a disjoint union

$$I_3(X, 0) = (\text{Sym}^3 X - \Delta) \cup (\mathbf{P}^2 \text{ bundle over } \Delta - \Delta_s) \cup (\mathbf{P}^5 \text{ bundle over } \Delta_s)$$

This leads to

$$\begin{aligned} e(I_3(X, 0)) &= \frac{1}{6}e(X^3 - \Delta) + 3e(X)(\Delta - \Delta_s) + 6e(X)(\Delta_s) \\ &= \frac{1}{6}(e(X)^3 - 3e(X)^2 + 2e(X)) + 3(e(X)^2 - e(X)) + 6e(X), \end{aligned}$$

which is immediately checked to be the coefficient of  $q^3$  in  $M(-q)^{e(X)}$ .

A similar and easier calculation, blowing up  $\text{Sym}^2 X$  along the diagonal shows that  $D_0^2$  is the coefficient of  $q^2$  in  $M(-q)^{e(X)}$ .

These techniques use the motivic property of the Euler characteristic. If the dimension zero Donaldson-Thomas invariants can be shown to be appropriately motivic, then Conjecture 1 would follow from the computation of the dimension zero Donaldson-Thomas invariants of  $\mathbf{C}^3$  as computed by localization.

A more general recipe for obtaining the component of  $I_n(X, 0)$  containing distinct points by blowing up a symmetric product of  $X$  is given in [ES].

## 2 GV invariants

From M-theory compactified on  $X$  we get conjectured Gopakumar-Vafa invariants  $n_\beta^g \in \mathbf{Z}$  [GV] which satisfy

$$F'(X) = \sum_{m, g, \beta \neq 0} n_\beta^g \frac{1}{m} \left( 2 \sin \frac{m\lambda}{2} \right)^{2g-2} t^{m\beta}. \quad (14)$$

At present, there is no intrinsic mathematical definition of the  $n_\beta^g$ . One can take (14) as a recursive definition of the Gopakumar-Vafa invariants in terms of the Gromov-Witten invariants, but then it is only clear that  $n_\beta^g \in \mathbf{Q}$ . The integrality of the  $n_\beta^g$  is called the *integrality conjecture*. A discussion from the mathematical perspective appears in [P2]. One can also attempt to define the GV invariants recursively using the DT invariants, as will be illustrated below. The equivalence of these recursive definitions would follow from the validity of Conjecture 2.

As is well known, Gromov-Witten invariants can be defined on noncompact Calabi-Yau threefolds in a number of interesting contexts, e.g. neighborhoods of super-rigid curves [P1] or homology classes contained in Fano surfaces.

**Examples.** Let  $X$  be the total space of the bundle  $\mathcal{O}(-1) \oplus \mathcal{O}(-1)$  over  $\mathbf{P}^1$  (“local  $\mathbf{P}^1$ ”). Identifying  $\mathbf{P}^1$  with the zero section we have  $n_{[\mathbf{P}^1]}^0 = 1$  and all other  $n_\beta^g = 0$ . For a general local elliptic curve we have  $n_{k[E]}^1 = 1$  for all  $k \geq 1$ , with all other  $n_\beta^g = 0$ . See [P1] for more detail.

Conjecture 2 together with (14) implies that the coefficient

$$\sum_m \frac{1}{m} \left( 2 \sin \frac{m\lambda}{2} \right)^{2g-2} t^{m\beta} \quad (15)$$

of a genus  $g > 0$  GV invariant in  $F'$  contributes to the DT partition function as

$$\begin{aligned} Z^{\text{DT}'} &= \exp \left( \sum_m \frac{1}{m} \left( 2 \sin \frac{m\lambda}{2} \right)^{2g-2} t^{m\beta} \right) \\ &= \exp \left( \sum_m \frac{1}{m} (-1)^{g-1} \left( e^{im\lambda/2} - e^{-im\lambda/2} \right)^{2g-2} t^{m\beta} \right) \\ &= \exp \left( \sum_{m,k} \frac{1}{m} (-1)^{k+g-1} \binom{2g-2}{k} e^{i(g-1-k)m\lambda} t^{m\beta} \right) \\ &= \exp \left( \sum_k (-1)^{k+g} \binom{2g-2}{k} \log \left( 1 - e^{i(g-1-k)\lambda} t^\beta \right) \right) \\ &= \prod_{k=0}^{2g-2} \left( 1 - e^{i(g-1-k)\lambda} t^\beta \right)^{(-1)^{k+g} \binom{2g-2}{k}} \\ &= \prod_{k=0}^{2g-2} \left( 1 + (-1)^{g-k} q^{g-1-k} t^\beta \right)^{(-1)^{k+g} \binom{2g-2}{k}}. \end{aligned} \quad (16)$$

If  $g = 0$ , we get

$$\begin{aligned}
Z^{\text{DT}'} &= \exp \left( \sum \frac{1}{m} \left( 2 \sin \frac{m\lambda}{2} \right)^{-2} t^{m\beta} \right) \\
&= \exp \left( \sum_m -e^{im\lambda} \frac{1}{m} \left( 1 - e^{im\lambda} \right)^{-2} t^{m\beta} \right) \\
&= \exp \left( \sum_{m,k} -\frac{1}{m} k e^{imk\lambda} t^{m\beta} \right) \\
&= \prod_{k=1}^{\infty} \left( 1 - e^{ik\lambda} t^{\beta} \right)^k \\
&= \prod_{k=1}^{\infty} \left( 1 + (-1)^{k+1} q^k t^{\beta} \right)^k.
\end{aligned} \tag{17}$$

Putting these two cases together, we get for  $Z^{\text{DT}}(X)'$  the product expression

$$\prod_{\beta} \left( \prod_{j=1}^{\infty} \left( 1 + (-1)^{j+1} q^j t^{\beta} \right)^{jn_{\beta}^0} \prod_{g=1}^{\infty} \prod_{k=0}^{2g-2} \left( 1 + (-1)^{g-k} q^{g-1-k} t^{\beta} \right)^{(-1)^{k+g} n_{\beta}^g \binom{2g-2}{k}} \right) \tag{18}$$

in terms of the GV invariants. We get  $Z^{\text{DT}}(X)$  by multiplying (18) by  $M(-q)^{e(X)}$ .

The factor corresponding to  $\beta$  in (18) can be expanded to

$$1 + t^{\beta} \left( n_{\beta}^g q^{1-g} + \left( n_{\beta}^{g-1} + (2g-2)n_{\beta}^g \right) + O(q^{3-g}) \right) + O(t^{2\beta}), \tag{19}$$

where  $g$  is now interpreted as the maximum  $g$  such that  $n_{\beta}^g$  is nonzero.

**Example.** The adaptation of [MNOP1, Theorem 3] to local  $\mathbf{P}^1$  combined with the computation of the Gromov-Witten invariants of local  $\mathbf{P}^1$  in [FP] gives

$$Z'_{\text{DT}} = \prod_{k=1}^{\infty} \left( 1 + (-1)^{k+1} q^k t^{\beta} \right)^k$$

as predicted. The Donaldson-Thomas invariants of local  $\mathbf{P}^1$  can also be computed by the methods of [ORV].

**Example.** If  $g = 1$ , then Conjecture 2 implies that  $Z'_{\text{DT}} = \prod_m (1 - t^{m\beta})^{-1} = \sum p(k) t^{k\beta}$ , where  $p$  is the classical partition function. We partially verify this prediction by computing  $D_{k[E]}^n$  for  $n = 0$  and all  $k$ , as well as for  $n = k = 1$ , where  $E \subset X$  is the elliptic curve.

It is straightforward to reduce to the case where  $X$  is locally  $L \oplus L^{-1}$ , with  $L$  a nontorsion degree zero line bundle on the elliptic curve  $E$ . Since

$e(X) = 0$ , it follows that the DT invariants and reduced DT invariants coincide in the local case. Note that  $L$  and  $L^{-1}$  are the only quotient bundles of  $\mathcal{I}_Z/\mathcal{I}_Z^2$  of degree 0. Let  $x$  and  $y$  denote the fiber coordinates on  $L$  and  $L^{-1}$  respectively, so that in particular the zero section  $E$  has equations  $x = y = 0$ . We can associate a monomial ideal in  $(x, y)$  to any partition of  $k$  in the usual way. In the current context, this monomial ideal defines a subscheme  $Z \subset X$  supported on  $E$ , with multiplicity  $k$  and  $\chi(\mathcal{O}_Z) = 0$ . It can be checked that these are the only such subschemes. Each of these ideals defines an isolated reduced point of  $I_0(X, k[E])$ . Since there are  $p(k)$  such points, this contributes  $p(k)t^{k\beta}$  to the DT partition function.

Now, returning to a more general  $X$ , suppose for simplicity that  $E$  is the only curve in  $X$  of class  $\beta$ . If  $p \in X - E$ , then  $Z = E \cup \{p\}$  has ideal sheaf  $\mathcal{I}_Z \in I_1(X, \beta)$ . There are also ideal sheaves  $\mathcal{I} \in I_1(X, \beta)$  with support  $E$  and a single embedded point. It is not hard to see from this that  $I_1(X, \beta) = \text{Bl}_E X$  (the proof is much easier than the proof of Lemma 1 below). Clearly  $e(\text{Bl}_E X) = e(X)$ , so  $D_\beta^1 = -e(X)$ . This implies

$$Z_\beta^{DT}(X) \equiv 1 - e(X)q \pmod{q^2}$$

which implies

$$Z_\beta^{DT}(X)' \equiv 1 \pmod{q^2},$$

consistent with the prediction  $Z_\beta^{DT}(X)' = 1$ .

In the context of DT invariants, we now revisit the methods for computing GV invariants which were developed in the physics literature [KKV].

Fix  $g \geq 0$ , and suppose  $I_{1-g}(X, \beta)$  parametrizes ideals of local complete intersection curves of arithmetic genus  $g$ .<sup>2</sup> To compare with [KKV], we put  $\mathcal{M} = I_{1-g}(X, \beta)$ . Identifying  $\mathcal{M}$  with a component of the Hilbert scheme, let  $\mathcal{C} \subset \mathcal{M} \times X$  be the universal subscheme, and let  $\mathcal{C}^{[n]}$  be the relative Hilbert scheme of  $n$  points in the family  $\mathcal{C}/\mathcal{M}$ . In particular  $\mathcal{C}^{[1]} = \mathcal{C}$ . Put  $\mathcal{C}^{[0]} = \mathcal{M}$ .

Fix  $\delta \leq g$ , and assume that  $\mathcal{C}^{[n]}$  is smooth for  $n \leq \delta$ . Then under certain additional hypotheses

**Conjecture 3** ([KKV])

$$(-1)^{\dim \mathcal{M} + \delta} n_\beta^{g-\delta} = e(\mathcal{C}^{[\delta]}) + (2g - 2\delta) e(\mathcal{C}^{[\delta-1]}) +$$

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<sup>2</sup>In [KKV], the curves were contained in smooth Fano surfaces embedded in Calabi-Yau threefolds, e.g. “local  $\mathbf{P}^2$ ”, but Proposition 2 below applies in greater generality.



$$\sum_{i=2}^{\delta} \frac{1}{i!} (2g - 2\delta + 2i - 2) (2g - 2\delta + i - 3) (2g - 2\delta + i - 4) \cdots (2g - 2\delta - 1) e(\mathcal{C}^{[\delta-i]}).$$

Using (19), Conjectures 2 and 3 imply that

$$D_{\beta}^{1-g'} = n_{\beta}^g = (-1)^{\dim \mathcal{M}} e(\mathcal{M}) \quad (20)$$

$$D_{\beta}^{2-g'} = n_{\beta}^{g-1} + (2g - 2)e(\mathcal{M}) = (-1)^{\dim \mathcal{C}} e(\mathcal{C}). \quad (21)$$

**Proposition 2** *In addition to the above hypotheses, suppose that all ideals  $\mathcal{I} \in I_{2-g}(X, \beta)$  have  $\mathcal{O}_X/\mathcal{I}$  supported on a curve parametrized by  $\mathcal{M}$ . Then, for  $\delta \leq 1$  the contribution of  $\mathcal{M}$  to  $Z_{\beta}^{\text{DT}}(X)'$  is*

$$\sum_{n=0}^{\delta} (-1)^{\dim \mathcal{C}^{[n]}} e(\mathcal{C}^{[n]}) q^{n+1-g} + O(q^{\delta+2-g}).$$

Proposition 2 has been written in this suggestive way as omitted computational evidence suggests that Conjecture 3 may imply the validity of the assertion of Proposition 2 with appropriate additional hypotheses for larger values of  $\delta$ . However, new techniques will be needed to investigate this situation, as the DT moduli spaces will not be smooth in general.

The examples in [KKV] satisfy the hypotheses of Proposition 2, which may therefore be viewed as providing a mathematically rigorous framework for the computational techniques developed in [KKV].

**Lemma 1** *Under the above hypotheses,  $I_{2-g}(X, \beta)$  is the blowup of  $\mathcal{M} \times X$  along  $\mathcal{C}$ . In particular,  $I_{2-g}(X, \beta)$  is smooth.*

*Proof:* Let  $\mathcal{I} \subset \mathcal{O}_{\mathcal{M} \times X}$  denote the universal ideal sheaf. Let  $\pi_{13} : \mathcal{M} \times X \times X \rightarrow \mathcal{M} \times X$  be the projection onto the first and third factors and consider the composition of mappings

$$\pi_{13}^* \mathcal{I} \hookrightarrow \mathcal{O}_{\mathcal{M} \times X \times X} \rightarrow \mathcal{O}_{\mathcal{M} \times \Delta_X} \quad (22)$$

with  $\Delta_X \subset X \times X$  the diagonal and the second map being the restriction. Interpreting (22) as a family of maps over  $\mathcal{M} \times X$ , its fiber over  $(Z, p)$  is surjective unless  $(Z, p) \in \mathcal{C}$ , so that the ideal of  $\mathcal{C}$  is defined as the scheme-theoretic locus in  $\mathcal{M} \times X$  over which (22) vanishes. Let  $\pi : \text{Bl}_{\mathcal{C}}(\mathcal{M} \times X) \rightarrow \mathcal{M} \times X$  be the blowup of  $\mathcal{C}$ , with  $E \subset \text{Bl}_{\mathcal{C}}(\mathcal{M} \times X)$  the exceptional divisor.

Let  $\iota : X \times X \rightarrow X \times X$  be the map interchanging factors of  $X$ , and let  $\rho$  denote the composition

$$\rho : \mathrm{Bl}_{\mathcal{C}}(\mathcal{M} \times X) \times X \xrightarrow{\pi \times 1_X} \mathcal{M} \times X \times X \xrightarrow{1_{\mathcal{M}} \times \iota} \mathcal{M} \times X \times X. \quad (23)$$

Then (22) induces a mapping

$$\phi : \rho^* \mathcal{I} \rightarrow \mathcal{O}_{\rho^{-1}(\mathcal{M} \times \Delta_X)}(-E). \quad (24)$$

Using local generators of  $\mathcal{I}$  it is straightforward to check that (24) is surjective. Locally over  $\mathcal{M} \times X$ , let  $\mathcal{I}$  be generated by  $f = f(Z, p, x)$ ,  $g = g(Z, p, x)$ , where  $Z \in \mathcal{M}$  and  $p, x \in X$ , so that  $\mathcal{C}$  is defined by  $f(Z, p, p) = g(Z, p, p) = 0$ . A local piece of the blowup is then given by introducing a new variable  $t$  subject to the relation  $f = tg$ . In this patch,  $E$  is defined by  $g(Z, p, p)$ .

Over  $(Z, p) \in \mathcal{M} \times X$ , (22) is given by

$$f \mapsto f(Z, p, p), \quad g \mapsto g(Z, p, p),$$

which pulls back in the blowup to

$$f \mapsto tg(Z, p, p), \quad g \mapsto g(Z, p, p).$$

Dividing by the local equation of  $E$  (24) is given by

$$f \mapsto t, \quad g \mapsto 1,$$

which is clearly surjective as claimed.

Let  $\tilde{\mathcal{I}}$  be the kernel of  $\phi$ . Clearly  $\tilde{\mathcal{I}}$  is flat since both terms in (24) are, and the fibers of  $\tilde{\mathcal{I}}$  over  $\mathrm{Bl}_{\mathcal{C}}(\mathcal{M} \times X)$  are ideal sheaves in  $I_{2-g}(X, \beta)$ . We claim that  $\tilde{\mathcal{I}} \subset \mathcal{O}_{\mathrm{Bl}_{\mathcal{C}}(\mathcal{M} \times X) \times X}$  is the universal ideal sheaf.

To check this, take an arbitrary family  $\mathcal{J} \subset \mathcal{O}_{S \times X}$  of ideal sheaves parameterized by a scheme  $S$ . The saturation  $\mathcal{J}'$  of  $\mathcal{J}$  is a family of ideals in  $I_{1-g}(X, \beta)$  and fits into a short exact sequence

$$0 \rightarrow \mathcal{J} \rightarrow \mathcal{J}' \xrightarrow{\alpha} \mathcal{O}_{\Gamma} \rightarrow 0, \quad (25)$$

where  $\Gamma$  is the graph of some morphism  $k : S \rightarrow X$ . The family of ideals  $\mathcal{J}'$  defines a morphism  $j : S \rightarrow \mathcal{M}$ , leading to a morphism

$$\psi : S \xrightarrow{j \times k} \mathcal{M} \times X$$

It remains to show that  $\psi$  factors through the blowup, inducing  $\mathcal{J}$  by pulling back  $\tilde{\mathcal{I}}$ . We must first show that the pullback of the ideal of  $\mathcal{C}$  to  $S$  is invertible. We again use local coordinates, letting  $f = f(s, x)$ ,  $g = g(s, x)$  generate  $\mathcal{J}'$  locally on  $S \times X$ , with  $s \in S$ ,  $x \in X$ . The pullback of the ideal  $I_{\mathcal{C}}$  of  $\mathcal{C}$  is generated by  $f(s, k(s))$  and  $g(s, k(s))$ . Fix  $s \in S$ . Without loss of generality, we can assume that the kernel of  $\alpha$  over  $s$  is generated by  $f$ , i.e.  $(\alpha(f))(s) = 0$ . Then  $f(s, k(s)) = 0$ . It follows that the pullback of  $I_{\mathcal{C}}$  to  $S$  is locally defined by the single equation  $g(s, k(s)) = 0$ . Hence  $j \times k$  factors through the blowup, giving the required map  $S \rightarrow \text{Bl}_{\mathcal{C}}(\mathcal{M} \times X)$ . Then the construction of  $\tilde{\mathcal{I}}$  combined with (25) shows that  $\tilde{\mathcal{I}}$  pulls back to  $\mathcal{J}$  as desired, QED.

*Proof of Proposition 2.* We have

$$I_{1-g}(X, \beta) = \mathcal{M}$$

which implies

$$D_{\beta}^{1-g} = (-1)^{\dim \mathcal{M}} e(\mathcal{M}).$$

Next

$$I_{2-g}(X, \beta) = \text{Bl}_{\mathcal{C}}(\mathcal{M} \times X)$$

which implies

$$D_{\beta}^{2-g} = (-1)^{\dim \mathcal{M}+1} (e(\mathcal{M})e(X) + e(\mathcal{C}))$$

by a simple topological argument analogous to the argument at the end of Section 1.2, noting that the exceptional divisor of the blowup is a  $\mathbf{P}^1$  bundle over  $\mathcal{C}$ . Thus the  $q$ -expansion of the DT partition function begins

$$Z_{\beta}^{DT} = (-1)^{\dim \mathcal{M}} e(\mathcal{M}) q^{1-g} + (-1)^{\dim \mathcal{M}+1} (e(\mathcal{M})e(X) + e(\mathcal{C})) q^{2-g} + O(q^{3-g}).$$

This leads immediately to

$$Z_{\beta}^{DT'} = Z_{\beta}^{DT} / Z_0^{DT} = (-1)^{\dim \mathcal{M}} e(\mathcal{M}) q^{1-g} + (-1)^{\dim \mathcal{M}+1} e(\mathcal{C}) q^{2-g} + O(q^{3-g}),$$

QED.

We close by explaining how an ad hoc argument in [KKV] becomes natural and even obvious in Donaldson-Thomas theory. A simple example will suffice.

Consider the computation of  $n_4^0 = -192$  in local  $\mathbf{P}^2$  from [KKV]. Naive application of the formula of Conjecture 3 gives the incorrect answer  $n_4^0 =$

−222. The explanation for the discrepancy of  $-222 - (-192) = -30$  was explained in [KKV] as arising from quartic curves which are unions of lines and cubics, i.e.  $n_1^0 n_3^1 = (3)(-10) = -30$ .

In Donaldson-Thomas theory, we would first compute the Donaldson-Thomas invariants and then use (18) to solve for the GV invariants. This illustrates the recursive procedure alluded to above. From (19) and the GV invariants in [KKV], we get factors for curves of degree  $d \leq 3$ :

$d$	DT factor
1	$1 + 3qt + \dots$
2	$1 - 6qt^2 + \dots$
3	$1 - 10t^3 + \dots$

(26)

In multiplying together the factors (19) for  $d \leq 3$  we get

$$1 - 30qt^4 + \dots \tag{27}$$

where the omitted terms are irrelevant to the computation (although not necessarily of higher order). The key point is that the  $d = 4$  factor (19) contains the term  $n_4^0 qt^4$ . So in computing the GV invariant, we divide  $Z_{\text{DT}}(X)'$  by (27) and equate the coefficient of  $qt^4$  with that of the  $d = 4$  factor, then solve for  $n_4^0$ . Thus the effect of the division is to subtract  $-30$  from the answer that would have otherwise been obtained without the contribution of curves of lower degree.

The conclusion is that the heuristic methods of [KKV] are in principle better adapted to the computation of DT invariants rather than GV invariants. In the sense above, the complications in [KKV] arise from solving for the GV invariants in terms of the DT invariants.

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